Lecture 6.

Thus far, we have constructions:

ρ on $\mathcal{E} \subseteq P(X)$ essentially arbitrary

$\downarrow$

μ^* outer measure s.t. $\mu^* \leq \rho$

$\downarrow$

$\mu = \mu^*|_{\mathcal{M}}$ complete meas. on $\mathcal{M} = \{\mu^*\text{-meas. sets}\}$

Would like $\mu = \rho$ on $\mathcal{E}$ ... but this cannot happen without conditions on $(\rho, \mathcal{E})$ (as we observed).

<u>Def 1.</u> Let $\mathcal{A} \subseteq P(X)$ be an algebra and $\mu_0 : \mathcal{A} \to [0,\infty]$ a function. Then, μ_0 is a <u>premeasure</u> if

(i) $\mu_0(\emptyset) = 0$

and

(ii) $\{A_k\}_{k=1}^{\infty}$ disjoint seq. of sets in $\mathcal{A}$ and $A = \bigcup_{k=1}^{\infty} A_k \in \mathcal{A}$ $\Longrightarrow$

not always true

$$\mu_0(A) = \sum_{k=1}^{\infty} \mu_0(A_k).$$

<u>Thm 1.</u> Let $\mathcal{A}$ be an algebra, μ_0 a premeasure of $\mathcal{A}$, and μ^* the outer measure given by

$$\mu^*(A) = \inf\left\{ \sum_{k=1}^{\infty} \mu_0(A_k) : A_k \in \mathcal{A}, A \subseteq \bigcup_{k=1}^{\infty} A_k \right\}$$

(a) $\mathcal{M}(\mathcal{A}) \subseteq \{\mu^*\text{-meas. sets}\} =: \mathcal{M}$ and the meas. $\mu = \mu^*|_{\mathcal{M}}$ satisfies $\mu = \mu_0$ on $\mathcal{A}$.

(b) If ν is another meas. s.t. $\nu = \mu_0$ on $\mathcal{A}$, then $\nu \leq \mu$ and $\nu(E) = \mu(E)$ for all $\mu(E) < \infty$.

(c) If μ_0 is σ-finite, then μ is unique.

Pf. (a) Recall from previous that the outer measure $\mu^* \le \mu_0$ on $\mathcal{A}$. For reverse ineq., pick $\varepsilon>0$, let $E\in\mathcal{A}$ and $E\subseteq\bigcup_{k=1}^{\infty}E_k$, $E_k\in\mathcal{A}$, such that

$$\mu^*(E) \ge \sum_{k=1}^{\infty}\mu_0(E_k)-\varepsilon.$$

Note that $F_k = E\cap\left(E_k\setminus\bigcup_{k=1}^{k-1}E_k\right)\in\mathcal{A}$, since $\mathcal{A}$ is an algebra. The F_k are disjoint and $E=\bigcup_{k=1}^{\infty}F_k$. By prop. of premeasure

$$\mu_0(E)=\sum_{k=1}^{\infty}\mu_0(F_k)\le\sum_{k=1}^{\infty}\mu_0(E_k)\le\mu^*(E)+\varepsilon.$$

Since $\varepsilon>0$ arbitrary $\Rightarrow \mu_0(E)\le\mu^*(E) \Rightarrow$ $\mu^*(E)=\mu_0(E)$.

We must also show that $\mathcal{M}(\mathcal{A})\subseteq\mathcal{M}:=\{\mu^*\text{-meas. sets}\}$ for which it suffices to show $\mathcal{A}\subseteq\mathcal{M}$. Thus, let $E\in\mathcal{A}$ and $B\subseteq X$.

Let $E_n \in \mathcal{A}$ s.t. $B \subseteq \bigcup_1^\infty E_n$ and

$$\mu^*(B) \geq \sum_1^\infty \mu_0(E_n) - \varepsilon \geq \sum_1^\infty \left(\mu_0(E_n \cap E) + \mu_0(E_n \cap E^c)\right) - \varepsilon \geq \mu^*(B \cap E) + \mu^*(B \cap E^c) - \varepsilon \Rightarrow \mu^*(B) \geq \mu^*(B \cap E) + \mu^*(B \cap E^c)$$

Since $\varepsilon > 0$ arb. $\Rightarrow$ $E \in \mathcal{M}$.

(b) Let $E \subseteq \mathcal{M}(\mathcal{A})$ and $A_n \in \mathcal{A}$ s.t. $E \subseteq \bigcup_1^\infty A_k$ and $\mu(E) \geq \sum_1^\infty \mu_0(A_n) - \varepsilon$.

By subadd., $\nu(E) \leq \sum_1^\infty \nu(A_k) = \sum^\infty \mu_0(A_n) \leq \mu(E) + \varepsilon \Rightarrow \boxed{\nu \leq \mu}$

Next, let $A = \bigcup_{n=1}^\infty A_n$. Then, by ,

$\mu(A) \leq \mu(E) + \varepsilon$, which is useless information if $\mu(E) = \infty$.

However, if $\mu(E)<\infty$, then this $\Rightarrow$ $\mu(A\setminus E)<\varepsilon$. Now, set $B_n=\bigcup_{k=1}^{n} A_k \in \mathcal{A}$.

$\Rightarrow$ $\mu(A)=\lim_{n\to\infty}\mu(B_n)=\lim_{n\to\infty}\mu_0(B_n)=$

$\lim_{n\to\infty}\nu(B_n)=\nu(A)$.

Then, $\mu(E)\le\mu(A)=\nu(A)=\nu(E)+$

$\nu(A\setminus E)\le\nu(E)+\mu(A\setminus E)=\nu(E)+\varepsilon$.

$\Rightarrow \mu(E)=\nu(E)$.

(c) If μ σ-finite, $X=\bigcup_{k=1}^{\infty} X_k$, $\mu(X_k)<\infty$. $\in \mathcal{M}(\mathcal{A})$

<u>WLOG</u>. $X_k\cap X_l=\emptyset$ if $k\neq l$. Thus, if $E\in\mathcal{M}(\mathcal{A})$, $E=\bigcup_{k=1}^{\infty} E\cap X_k$ and

$$\mu(E)=\sum_{k=1}^{\infty}\mu(E\cap X_k)=\sum_{k=1}^{\infty}\nu(E\cap X_k)=\nu(E).$$

$<\infty$ by (b)